Chapter 19. Acousto-Optics

19.1 Interaction of Light and Sound
19.2 Acousto-Optic Devices
19.3 Acousto-Optics of Anisotropic Media
Acousto-Optic effect:
The refractive index of an optical medium is altered by the presence of sound.

⇒ Light can be controlled by sound~

Applications:
Optical Modulators, Switches, Filters, Isolators, Frequency shifters, Spectrum analyzers, …

Change in refractive index of material by sound wave:
- Gas: Density modification by dynamic strain (sound wave)
- Solid, Liquid: Alternation of the optical polarizability by the vibrations of molecules
An acoustic wave creates a perturbation of the refractive index in the form of a wave. Therefore, the medium becomes a “dynamic” graded-index medium (an inhomogeneous medium with a time-varying stratified refractive index). The theory of acousto-optics deals with the perturbation of the refractive index, and with the propagation through the perturbed time-varying inhomogeneous medium.

However, since optical frequencies are much greater than acoustic frequencies. As a consequence, it is possible to use an adiabatic approach in which the optical propagation problem is solved separately at every instant of time, always treating the medium as if it were a static inhomogeneous medium. In this quasi-stationary approximation, acousto-optics becomes the optics of an inhomogeneous medium (usually periodic) that is controlled by sound.
19. 1 Interaction of light and sound

A. Bragg Diffraction

Considering an acoustic plane wave traveling in the $x$ direction, the strain is

$$s(x, t) = S_0 \cos(\Omega t - qx), \quad (19.1-1)$$

where, $\Omega = 2\pi f$ : angular frequency, $q = 2\pi/\Lambda$ : wavenumber

Acoustic intensity: $I_s = \frac{1}{2} \varrho v_s^3 S_0^2$, \hfill (19.1-2)

Refractive index change is given by [analogous to the Pockels effect in (20.1-4)]

$$\Delta n(x, t) = -\frac{1}{2} p n^3 s(x, t), \quad (19.1-3)$$

where, $p$ : Photoelastic constant (strain-optic coefficient)
Consequently, refractive index of medium is

\[ n(x, t) = n - \Delta n_0 \cos(\Omega t - qx) \]  \hspace{1cm} (19.1-4)

where, \( \Delta n_0 = \frac{1}{2} p n^3 S_0 \).  \hspace{1cm} (19.1-5)

Substituting from (19.1-2) into (19.1-5),

\[ \Delta n_0 = \sqrt{\frac{1}{2} M I_s} \propto \sqrt{I_s} \]  \hspace{1cm} (19.1-6)

where, \( M = \frac{p^2 n^6}{\varrho v_s^3} \): Figure of merit for the strength of the AO effect in the material.

**EXAMPLE 19.1-1. Acousto-optic Effect Figure of Merit.** In extra-dense flint glass \( \varrho = 6.3 \times 10^3 \text{ kg/m}^3 \), \( v_s = 3.1 \text{ km/s} \), \( n = 1.92 \), \( p = 0.25 \), so that \( M = 1.67 \times 10^{-14} \text{ m}^2/\text{W} \). An acoustic wave of intensity 10 W/cm\(^2\) creates a refractive-index wave of amplitude \( \Delta n_0 = 2.89 \times 10^{-5} \).
Table 9.2. Strain–Optic Coefficients* [1]

(a) Isotropic System

<table>
<thead>
<tr>
<th>Substance</th>
<th>Wavelength $\lambda$ (µm)</th>
<th>$P_{11}$</th>
<th>$P_{12}$</th>
<th>$P_{44}$</th>
<th>$P_{11} - P_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fused silica (SiO$_2$)</td>
<td>0.63</td>
<td>0.121</td>
<td>0.270</td>
<td></td>
<td></td>
</tr>
<tr>
<td>As$_2$S$_3$ glass</td>
<td>1.15</td>
<td>0.308</td>
<td>0.299</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Water</td>
<td>0.63</td>
<td>±0.31</td>
<td>±0.31</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ge$_3$S$_2$As$_2$ (glass)</td>
<td>1.06</td>
<td>±0.21</td>
<td>±0.21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lucite</td>
<td>0.63</td>
<td>±0.30</td>
<td>±0.28</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Polystyrene</td>
<td>0.63</td>
<td>±0.30</td>
<td>±0.31</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) Cubic System: Classes 43m, 432, and m3m

<table>
<thead>
<tr>
<th>Substance</th>
<th>Wavelength $\lambda$ (µm)</th>
<th>$P_{11}$</th>
<th>$P_{12}$</th>
<th>$P_{44}$</th>
<th>$P_{11} - P_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CdTe</td>
<td>10.60</td>
<td>-0.152</td>
<td>-0.017</td>
<td>-0.057</td>
<td>-0.135</td>
</tr>
<tr>
<td>GaAs</td>
<td>1.15</td>
<td>-0.165</td>
<td>-0.140</td>
<td>-0.072</td>
<td>-0.025</td>
</tr>
<tr>
<td>GaP</td>
<td>0.633</td>
<td>-0.151</td>
<td>-0.082</td>
<td>-0.074</td>
<td>-0.069</td>
</tr>
<tr>
<td>Ge</td>
<td>2.0–2.2</td>
<td>0.063</td>
<td>-0.0535</td>
<td>-0.074</td>
<td>-0.0095</td>
</tr>
<tr>
<td></td>
<td>10.60</td>
<td>0.27</td>
<td>0.235</td>
<td>0.125</td>
<td></td>
</tr>
<tr>
<td>NaCl</td>
<td>0.55–0.65</td>
<td>0.115</td>
<td>0.159</td>
<td>-0.011</td>
<td>-0.042</td>
</tr>
<tr>
<td>NaF</td>
<td></td>
<td>0.633</td>
<td>0.08</td>
<td>0.20</td>
<td>-0.03</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0.589</td>
<td></td>
<td></td>
<td>-0.12</td>
</tr>
<tr>
<td>Si</td>
<td>1.15</td>
<td>-0.101</td>
<td>0.0094</td>
<td>-0.110</td>
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</tr>
<tr>
<td></td>
<td>3.39</td>
<td>-0.094</td>
<td>0.017</td>
<td>-0.051</td>
<td>-0.111</td>
</tr>
<tr>
<td>Y$_2$Fe$_2$O$_4$</td>
<td>1.15</td>
<td>±0.025</td>
<td>±0.073</td>
<td>±0.041</td>
<td></td>
</tr>
<tr>
<td>Y$_2$Al$_2$O$_12$</td>
<td>0.633</td>
<td>-0.029</td>
<td>0.0091</td>
<td>-0.0615</td>
<td>-0.038</td>
</tr>
<tr>
<td>KRS5</td>
<td>0.633</td>
<td>±0.18</td>
<td>±0.27</td>
<td>±0.15</td>
<td></td>
</tr>
<tr>
<td>KRS6</td>
<td>0.633</td>
<td>±0.28</td>
<td>±0.25</td>
<td>±0.14</td>
<td></td>
</tr>
<tr>
<td>$\beta$-ZnS</td>
<td>0.546</td>
<td></td>
<td></td>
<td>-0.044</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.589</td>
<td></td>
<td></td>
<td>-0.137</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.633</td>
<td>0.091</td>
<td>-0.01</td>
<td>0.075</td>
<td>0.010</td>
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### Table 9.2. (Continued)

(c) Hexagonal System: Classes 6m2, 6mm, 622, and 6/mmm

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<tr>
<th>Substance</th>
<th>Wavelength $\lambda$ (µm)</th>
<th>$p_{11}$</th>
<th>$p_{12}$</th>
<th>$p_{13}$</th>
<th>$p_{31}$</th>
<th>$p_{33}$</th>
<th>$p_{44}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CdS</td>
<td>0.633</td>
<td>-0.142</td>
<td>-0.066</td>
<td>-0.057</td>
<td>-0.041</td>
<td>-0.20</td>
<td>± 0.054</td>
</tr>
<tr>
<td></td>
<td>10.60</td>
<td></td>
<td>0.104</td>
<td></td>
<td>0.011</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SnO</td>
<td>0.633</td>
<td>± 0.222</td>
<td>± 0.099</td>
<td>-0.111</td>
<td>± 0.088</td>
<td>-0.235</td>
<td>-0.0585</td>
</tr>
<tr>
<td>$\alpha$-ZnS</td>
<td>0.633</td>
<td>-0.115</td>
<td>0.017</td>
<td>0.025</td>
<td>0.0271</td>
<td>-0.13</td>
<td>-0.0627</td>
</tr>
<tr>
<td>(wurtzite)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(d) Trigonal System: Classes 3m, 32, and 3m

<table>
<thead>
<tr>
<th>Substance</th>
<th>Wavelength $\lambda$ (µm)</th>
<th>$p_{11}$</th>
<th>$p_{12}$</th>
<th>$p_{13}$</th>
<th>$p_{14}$</th>
<th>$p_{31}$</th>
<th>$p_{33}$</th>
<th>$p_{41}$</th>
<th>$p_{44}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Al}_2\text{O}_3$</td>
<td>0.644</td>
<td>-0.23</td>
<td>-0.03</td>
<td>0.02</td>
<td>0.00</td>
<td>-0.04</td>
<td>-0.20</td>
<td>0.01</td>
<td>-0.10</td>
</tr>
<tr>
<td>$\text{LiNbO}_3$</td>
<td>0.633</td>
<td>-0.026</td>
<td>0.090</td>
<td>0.133</td>
<td>-0.075</td>
<td>0.179</td>
<td>0.071</td>
<td>-0.151</td>
<td>0.146</td>
</tr>
<tr>
<td>$\text{LiTaO}_3$</td>
<td>0.633</td>
<td>-0.081</td>
<td>0.081</td>
<td>0.093</td>
<td>-0.026</td>
<td>0.089</td>
<td>-0.044</td>
<td>-0.085</td>
<td>0.028</td>
</tr>
<tr>
<td>$\text{SiO}_2$ (quartz)</td>
<td>0.589</td>
<td>0.16</td>
<td>0.27</td>
<td>0.27</td>
<td>-0.030</td>
<td>0.29</td>
<td>0.10</td>
<td>-0.047</td>
<td>-0.079</td>
</tr>
<tr>
<td>$\text{Ag}_3\text{AsS}_3$ (proustite)</td>
<td>0.633</td>
<td>± 0.10</td>
<td>± 0.19</td>
<td>± 0.22</td>
<td>± 0.24</td>
<td>± 0.20</td>
<td>± 0.20</td>
<td>± 0.103</td>
<td>± 0.01</td>
</tr>
<tr>
<td></td>
<td>1.15</td>
<td>± 0.056</td>
<td>± 0.082</td>
<td>± 0.068</td>
<td>± 0.103</td>
<td>± 0.100</td>
<td>± 0.01</td>
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<td></td>
</tr>
</tbody>
</table>
### (e) Tetragonal System: Classes 4mm, 42m, 422, and 4/mmm

<table>
<thead>
<tr>
<th>Substance</th>
<th>Wavelength $\lambda$ ($\mu$m)</th>
<th>$p_{11}$</th>
<th>$p_{12}$</th>
<th>$p_{13}$</th>
<th>$p_{31}$</th>
<th>$p_{33}$</th>
<th>$p_{44}$</th>
<th>$p_{66}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\text{NH}_4)_2\text{H}_2\text{PO}_4$ (ADP)</td>
<td>0.589</td>
<td>0.319</td>
<td>0.277</td>
<td>0.169</td>
<td>0.197</td>
<td>0.167</td>
<td>-0.058</td>
<td>-0.091</td>
</tr>
<tr>
<td>MgF$_2$</td>
<td>0.546</td>
<td>$p_{11} - p_{12} = 0.892, p_{11} - p_{31} = 0.0695, p_{33} - p_{31} = 0.1128$</td>
<td>0.0776</td>
<td>0.0448</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\text{KH}_2\text{PO}_4$ (KDP)</td>
<td>0.589</td>
<td>0.287</td>
<td>0.282</td>
<td>0.174</td>
<td>0.241</td>
<td>0.122</td>
<td>-0.019</td>
<td>-0.064</td>
</tr>
<tr>
<td>$\text{Sr}<em>x\text{Ba}</em>{1-x}\text{Nb}_2\text{O}_6$:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x = 0.75$</td>
<td>0.63</td>
<td>0.16</td>
<td>0.10</td>
<td>0.08</td>
<td>0.11</td>
<td>0.47</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x = 0.5$</td>
<td>0.633</td>
<td>0.06</td>
<td>0.08</td>
<td>0.17</td>
<td>0.09</td>
<td>0.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TeO$_2$</td>
<td>0.633</td>
<td>0.0074</td>
<td>0.187</td>
<td>0.340</td>
<td>0.0905</td>
<td>0.240</td>
<td>-0.17</td>
<td>-0.0463</td>
</tr>
<tr>
<td>TiO$_2$ (rutile)</td>
<td>0.514</td>
<td>-0.001</td>
<td>0.113</td>
<td>-0.167</td>
<td>-0.106</td>
<td>-0.064</td>
<td>0.0095</td>
<td>-0.066</td>
</tr>
<tr>
<td></td>
<td>0.633</td>
<td>-0.011</td>
<td>0.172</td>
<td>-0.168</td>
<td>-0.0965</td>
<td>-0.058</td>
<td>± 0.072</td>
<td></td>
</tr>
</tbody>
</table>

### (f) Tetragonal System: Classes 4, 4, and 4/m

<table>
<thead>
<tr>
<th>Substance</th>
<th>Wavelength $\lambda$ ($\mu$m)</th>
<th>$p_{11} = 0.24$</th>
<th>$p_{12} = 0.24$</th>
<th>$p_{13} = 0.0255$</th>
<th>$p_{16} = 0.017$</th>
<th>$p_{31} = 0.175$</th>
<th>$p_{33} = 0.300$</th>
<th>$p_{44} = 0.067$</th>
<th>$p_{45} = -0.01$</th>
<th>$p_{61} = 0.013$</th>
<th>$p_{66} = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>PbMoO$_4$</td>
<td>0.633</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>CdMoO$_4$</td>
<td>0.633</td>
<td>$p_{11} = 0.12$</td>
<td>$p_{12} = 0.10$</td>
<td>$p_{13} = 0.13$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Nonlinear Optics Lab.  Hanyang Univ.
<table>
<thead>
<tr>
<th>Material</th>
<th>$k_e$ ($\text{mg/m}^3$)</th>
<th>$\rho \times 10^{-3}$</th>
<th>$c_e$ (km/s)</th>
<th>$n$</th>
<th>$p$</th>
<th>$M_\omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water</td>
<td>1.0</td>
<td>1.5</td>
<td>1.33</td>
<td>0.31</td>
<td>1.0</td>
<td></td>
</tr>
<tr>
<td>Extra-dense flint glass</td>
<td>6.3</td>
<td>3.1</td>
<td>1.92</td>
<td>0.25</td>
<td>0.12</td>
<td></td>
</tr>
<tr>
<td>Fused quartz (SiO$_2$)</td>
<td>2.2</td>
<td>5.97</td>
<td>1.46</td>
<td>0.20</td>
<td>0.006</td>
<td></td>
</tr>
<tr>
<td>Polystyrene</td>
<td>1.06</td>
<td>2.35</td>
<td>1.59</td>
<td>0.31</td>
<td>0.8</td>
<td></td>
</tr>
<tr>
<td>KRS-5</td>
<td>7.4</td>
<td>2.11</td>
<td>2.60</td>
<td>0.21</td>
<td>1.6</td>
<td></td>
</tr>
<tr>
<td>Lithium niobate (LiNbO$_3$)</td>
<td>4.7</td>
<td>7.40</td>
<td>2.25</td>
<td>0.15</td>
<td>0.012</td>
<td></td>
</tr>
<tr>
<td>Lithium fluoride (LiF)</td>
<td>2.6</td>
<td>6.00</td>
<td>1.39</td>
<td>0.13</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>Rutile (TiO$_2$)</td>
<td>4.26</td>
<td>10.30</td>
<td>2.60</td>
<td>0.05</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>Sapphire (Al$_2$O$_3$)</td>
<td>4.0</td>
<td>11.00</td>
<td>1.76</td>
<td>0.17</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>Lead molybdate (PbMoO$_4$)</td>
<td>6.95</td>
<td>3.75</td>
<td>2.30</td>
<td>0.28</td>
<td>0.22</td>
<td></td>
</tr>
<tr>
<td>Alpha iodic acid (HIO$_3$)</td>
<td>4.63</td>
<td>2.44</td>
<td>1.90</td>
<td>0.41</td>
<td>0.5</td>
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</tr>
<tr>
<td>Tellurium dioxide (TeO$_2$)</td>
<td>5.99</td>
<td>0.617</td>
<td>2.35</td>
<td>0.09</td>
<td>5.0</td>
<td></td>
</tr>
<tr>
<td>(Slow shear wave)</td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

*Figure of merit of the material relative to water*
Consider an optical plane wave traveling through the acousto-optic medium,

Refractive index can be regarded as a static “frozen” sinusoidal function (Adiabatic approach),

\[ n(x) = n - \Delta n_0 \cos(qx - \varphi), \]

where, \( \varphi = \Omega t \) : fixed phase
Amplitude reflection

Divide the medium into incremental planar layers orthogonal to the $x$ axis, since the optical plane wave is partially reflected at each layer because of the refractive-index change. And, assume that the reflectance is sufficiently small (transmitted light is not depleted),

Total complex amplitude reflectance:

$$ r = \int_{-L/2}^{L/2} e^{j2kx \sin \theta} \frac{dr}{dx} dx. $$

(19.1-9)

Included since the reflected wave at $x$ is advanced by a distance $2x \sin \theta$, corresponding to a phase shift $2kx \sin \theta$, relative to the reflected wave at $x=0$. (refer to p.178, “Intro. to Optics”, Pedrotti)

$$ \frac{dr}{dx} = \frac{dr}{dn} \frac{dn}{dx} = \frac{dr}{dn} q \Delta n_0 \sin(qx - \varphi), $$

(19.1-10)

Independent of $x$
Using complex notation, \( \sin(qx - \varphi) = [e^{ij(qx-\varphi)} - e^{-ij(qx-\varphi)}] \)

\[
r = j\tau_0 e^{i\varphi} \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{i(2k \sin \theta - q)x} \, dx - j\tau_0 e^{-i\varphi} \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} e^{i(2k \sin \theta + q)x} \, dx,
\]

where, \( \tau_0 = \frac{1}{2} \Delta n_0 q L \frac{d\tau}{dn} \).  

(19.1-11)

Performing the integral, and substituting \( \varphi = \Omega t \),

\[
r = r_+ + r_-,
\]

(19.1-13)

where, \( r_\pm = \pm j\tau_0 \text{sinc} \left[ \frac{(2k \sin \theta \mp q)L}{2\pi} \right] e^{\pm j\Omega t} \)  

(19.1-14)

\( r^+, r^- \): upshifted and downshifted reflections, respectively

\( L \) is sufficiently large, two maxima at \( \theta = \pm \sin^{-1}(q/2k) \) are sharp, so that any slight deviation from the angles makes the counter term negligible. Thus, only one of these two term is significant at a time.
**Bragg condition** *(upshifted reflection)*

Sinc function has maximum when the argument is zero: \(2k \sin \theta - q = 0\) (for upshifted reflection)

\[
\sin \theta_B = \frac{\lambda}{2\Lambda}
\]

(19.1-15)  
**Bragg Angle**

Bragg angle is the angle for which the incremental reflection from planes separated by an acoustic wavelength \(\Lambda\) have a phase shift of \(2\pi\)  
\((2kx \sin \theta = 2k\Lambda \cdot (\lambda/2\Lambda) = 2\pi)\) so that they interfere constructively.

**EXAMPLE 19.1-2. Bragg Angle.** An acousto-optic cell is made of flint glass in which the sound velocity is \(v_s = 3\) km/s and the refractive index is \(n = 1.95\). The Bragg angle for reflection of an optical wave of free-space wavelength \(\lambda_o = 633\) nm \((\lambda = \lambda_o/n = 325\) nm) from a sound wave of frequency \(f = 100\) MHz \((\Lambda = v_s/f = 30\) \(\mu\)m) is \(\theta_B = 5.4\) mrad \(\approx 0.31^\circ\). This angle is internal (i.e., inside the medium). If the cell is placed in air, \(\theta_B\) corresponds to an external angle \(\theta'_B \approx n\theta_B = 0.61^\circ\). A sound wave of 10 times greater frequency \((f = 1\) GHz\) corresponds to a Bragg angle \(\theta_B = 3.1^\circ\).
Bragg condition \((q = 2k \times \sin \theta_B)\) is equivalent to the vector relation,

\[
k_r = k + q \quad (19.1-16)
\]

\[
q = (q, 0, 0), \; k = (-k \sin \theta, 0, k \cos \theta), \; \text{and} \; k_r = (k \sin \theta, 0, k \cos \theta)
\]

**Figure 19.1-2** The Bragg condition \(\sin \theta_B = q/2k\) is equivalent to the vector relation \(k_r = k + q\).
**Tolerance in the Bragg condition**

\[ 2k \sin \theta_B = q \]

(19.1-14)

\[ \Rightarrow \quad \text{sinc}[(2k\sin \theta - q)L / 2\pi] = \text{sinc}[2k(\sin \theta - \sin \theta_B)L / 2\pi] \]

\[ = \text{sinc}[(\sin \theta - \sin \theta_B)2L / \lambda] \]

First zero at \( \sin \theta - \sin \theta_B = \lambda / 2L \) \( \Rightarrow \quad \theta - \theta_B \approx \lambda / 2L \).

Since \( L \) is typically much greater than \( \lambda \). This is an extremely small angular width.
**Doppler shift**

(19.1-14) \( r^+ \propto \exp(j\Omega t) \)

Incident light field is proportional to \( \exp(j\omega t) \) \([E \propto \exp(j\omega t)]\), so the reflected wave has the form of \( E_{r^+} \propto r^+ E \propto \exp(j(\omega + \Omega)t) \), and has angular frequency

\[
\omega_r = \omega + \Omega \quad \text{(19.1-17)}
\]

Doppler Shift

: The frequency as well as the propagation direction of the reflected wave is changed.

The frequency shift equal to the frequency of sound.

\( \Rightarrow \) This can almost be thought as a Doppler shift.

Doppler shifted frequency:

\[
\omega_r = \omega\left(1 + 2v_s \sin \theta / c \right)
\]

\[
\sin \theta = \lambda / 2\Lambda, \ v_s = \Lambda \Omega / 2\pi, \ \text{and} \ c = \lambda \omega / 2\pi
\]

\[
\therefore \omega_r = \omega\left[1 + 2(\Lambda \Omega / 2\pi)(\lambda / 2\Lambda)(2\pi / \lambda \omega)\right] = \omega + \Omega
\]
**Peak reflectance (skip)**

Reflectance $\mathcal{R} = |r_+|^2$ at the Bragg angle $\theta = \theta_B$ is [from (19.1-12)]

$$\mathcal{R} = \frac{1}{4} \Delta n_0^2 q^2 L^2 \left| \frac{dr}{dn} \right|^2.$$  \hspace{1cm} (19.1-18)

From (6.2-8), (6.2-9),

$$t_x = 1 + r_x,$$

$$r_x = \frac{n_1 \cos \theta_1 - n_2 \cos \theta_2}{n_1 \cos \theta_1 + n_2 \cos \theta_2},$$

$$r_y = \frac{n_1 \sec \theta_1 - n_2 \sec \theta_2}{n_1 \sec \theta_1 + n_2 \sec \theta_2},$$

$$t_y = (1 + r_y) \frac{\cos \theta_1}{\cos \theta_2}. \hspace{1cm} (6.2-8) \hspace{1cm} (6.2-9)$$

$T_E$ Polarization

$T_M$ Polarization

Fresnel Equations

$n_1 \rightarrow n + \Delta n$, $n_2 = n$, $\theta_1 = 90^\circ - \theta$, $n_1 \sin \theta_1 = n_2 \sin \theta_2$, and neglect term of 2nd order in $\Delta n$.

$$\frac{dr}{dn} = \frac{-1}{2 n \sin^2 \theta}.$$  \hspace{1cm} (TE)  \hspace{1cm} (19.1-19)

$$\frac{dr}{dn} = \frac{-\cos 2\theta}{2 n \sin^2 \theta}.$$  \hspace{1cm} (TM)  \hspace{1cm} (19.1-20)

For small $\theta$, $\cos 2\theta \approx 1$.
From (19.1-18), by using \( q = 2k \sin \theta = (4\pi n \sin \theta / \lambda_o) \),

\[
R = \frac{\pi^2}{\lambda_o^2} \left( \frac{L}{\sin \theta} \right)^2 \Delta n_0^2. 
\]

or, by (19.1-6)

\[
\sin \theta = \frac{\lambda}{2\Lambda}, \quad R = \frac{2\pi^2}{\lambda_o^2} \left( \frac{L}{\sin \theta} \right)^2 MI_s
\]

- \( R \propto \lambda_0^{-4} \propto \omega^4 \): typical light-scattering phenomena.

- \( R \propto I_s \): valid only low intensity region since this result is obtained from the approximation theory (19.1B: Coupled-wave theory)
Downshifted Bragg Diffraction

\[ 2k \sin \theta_B = -q \]

\[ r_\pm = -j r_0 e^{-j \Omega t}. \]

\[ \omega_s = \omega - \Omega \]

\[ k_s = k - q, \]

(19.1-25)

(19.1-26)

(19.1-27)

Figure 19.1-5 Geometry of downshifted reflection of light from sound. The frequency of the reflected wave is downshifted.
**Quantum interpretation**

Photon \((\omega, k)\) interacts with acoustic phonon \((\Omega, q)\), and generates a new reflected photon \((\omega_r, k)\).

Energy and momentum conservation conditions:

\[
\begin{align*}
\hbar \omega_r &= \hbar \omega + \hbar \Omega \\
\hbar k_r &= \hbar k + \hbar q
\end{align*}
\]

\(\omega_r = \omega + \Omega\)

\(k_r = \vec{k} + \vec{q}\)
B. Coupled-wave theory

**Bragg diffraction as a scattering process**

Wave equation in a homogeneous medium with a slowly varying inhomogeneous refractive-index perturbation $\Delta n$, (5.2-20)

\[
\nabla^2 \mathcal{E} - \frac{1}{c^2} \frac{\partial^2 \mathcal{E}}{\partial t^2} \approx -S, \quad (19.1-28)
\]

where,

\[
S = -\mu_o \frac{\partial^2 \Delta \rho}{\partial t^2} = -2\mu_o \varepsilon_o n \frac{\partial^2}{\partial t^2} (\Delta n \mathcal{E}), \quad (19.1-29)
\]

Radiation Source Term
First Born approximation:

Assume that the scattering source $S$ is induce by the incident field [not by the actual (local) field].

- Incident field (plane wave),
  \[ E = \text{Re}\{A \exp[j(\omega t - k \cdot r)]\} \quad (19.1-30) \]

- Perturbation in $n$ caused by acoustic wave (plane wave),
  \[ \Delta n = -\Delta n_0 \cos(\Omega t - q \cdot r), \quad (19.1-31) \]

\[ S = -\left(\frac{\Delta n_0}{n}\right) (k_r^2 \text{Re}\{A \exp[j(\omega_r t - k_r \cdot r)]\} + k_s^2 \text{Re}\{A \exp[j(\omega_s t - k_s \cdot r)]\}) \]

(19.1-32)

where $\omega_r = \omega + \Omega$, $k_r = k + q$, $k_r = \omega_r/c$; and $\omega_s = \omega - \Omega$, $k_s = k - q$, $k_s = \omega_s/c$
Coupled-wave equations (upshifted Bragg diffraction)

\[ \mathcal{E} = \text{Re}\{E \exp(j\omega t)\} + \text{Re}\{E_r \exp(j\omega_r t)\} \]

\[ \Delta n = -\Delta n_0 \cos(\Omega t - \mathbf{q} \cdot \mathbf{r}) \]

\[ S = \text{Re}\{S \exp(j\omega t) + S_r \exp(j\omega_r t)\} + \text{terms of other frequencies}, \quad (19.1-33) \]

where,

\[ S = -k^2 \frac{\Delta n_0}{n} E_r, \quad S_r = -k_{\omega r}^2 \frac{\Delta n_0}{n} E. \quad (19.1-34) \]

(19.1-28) ➔

\[ (\nabla^2 + k^2)E = -S, \quad (\nabla^2 + k_{\omega r}^2)E_r = -S_r. \quad (19.1-35) \]

Refer to Appendix II ~…
Small-angle Bragg diffraction

For the case of small-angle diffraction (θ << 1), two waves travel approximately in the z direction:

\[ E = A \exp(-j k z) \text{ and } E_r = A_r \exp(-j k z) \]

\[ (\nabla^2 + k^2)A \exp(-j k z) \approx -j 2k (dA/dz) \exp(-j k z) \]

(19.1-35)

Boundary condition: \( A_r(0) = 0 \)

\[ A(z) = A(0) \cos \frac{\gamma z}{2} \]  
\[ A_r(z) = j A(0) \sin \frac{\gamma z}{2} \]

(19.1-38a, 19.1-38b)
Reflectance:

\[ R_e = \frac{|A_r(d)|^2}{|A(0)|^2} = \sin^2(\gamma d/2) \]

\[ R = \frac{\pi^2}{\lambda_0^2} \left( \frac{L}{\sin \theta} \right)^2 \Delta n_0^2 \quad (19.1-21) \]

\[ \gamma = \frac{k}{n} \Delta n_0 \quad (19.1-37) \]
C. Bragg diffraction of beams

**Diffraction of an optical beam from an acoustic plane wave**

For an optical beam of width $D$ interacting with an acoustic plane wave, the optical beam can be decomposed into plane waves with directions occupying a cone of half-angle, $\delta \theta = \frac{\lambda}{D}$.

- **Rectangular**: $1 \frac{\lambda}{D}$
- **Circular**: $1.22 \frac{\lambda}{D}$
- **Gaussian** (waist diameter $= 2w_0$): $\frac{\lambda}{\pi w_0} = 0.64 \frac{\lambda}{D}$

Although there is only one wave-vector $q$, there are many wave-vectors $k$ within a cone angle. But there is only one direction $k$ for which the Bragg condition is satisfied.

⇒ The reflected wave is plane wave with only one wave-vector $k_r$. 
Diffraction of an optical beam from an acoustic beam

Suppose now that the acoustic wave itself is a beam of width $D_s$. Then angular divergence,

$$\delta \theta_s = \frac{\Lambda}{D_s}.$$  \hspace{1cm} (19.1-40)

If the acoustic beam divergence is greater than the optical beam divergence ($\delta \theta_s \gg \delta \theta$), and if the central direction of two beams satisfy the Bragg condition, every optical plane wave find acoustic match and the reflected light beam has the same divergence as the incident beam.

**Figure 19.1-9** Diffraction of an optical beam from a sound beam.
**Diffraction of an optical plane wave from a thin acoustic beam: Raman-Nath Diffraction**

**Figure 19.1-10** An optical plane wave incident normally on a thin-beam acoustic standing wave is partially deflected into two directions making angles $\approx \pm \lambda/\Lambda$.

\[
\sin \frac{\theta}{2} = \frac{\lambda}{2\Lambda}.
\]  
(19.1-41)

\[
\theta \approx \frac{\lambda}{\Lambda}
\]  
(19.1-42)
Thin acoustic beam act as a **thin phase (diffraction) grating**.

![Diagram of thin acoustic beam](image)

**Quantum interpretation:**
One incident photon combines with two phonons to form a photon of second-order reflected wave. Conservation of momentum condition is $k_r = k \pm 2q$. The energy conservation condition is $\omega_r = \omega \pm 2\Omega$. Similar interpretations apply to higher orders.

Refer to Appendix III~
A. Modulators

Using an electrically controlled acoustic transducer (PZT), the intensity of the reflected light can be varied. ➔ can be used as a modulator or switch.

\[ R_e = \sin^2 \left( \frac{\pi \Delta n_0 d}{\lambda_0} \right) \]

(19.1.B), Reflectance, \( R_e \) can be unity: 100% reflection ➔ Efficient frequency shifter
Modulation bandwidth

Bandwidth: Maximum frequency range (band) in which modulator can efficiently modulate.

[In idealized condition]
When both the optical and the acoustic waves are plane waves, the frequency of sound corresponds to a Bragg angle,

\[ \theta = \sin^{-1} \frac{\lambda}{2\Delta} = \sin^{-1} \frac{f\lambda}{2v_s} \approx \frac{\lambda}{2v_s} f \]

(19.2-1)

For a fixed angle of incidence \( \theta \), an incident monochromatic optical plane wave of wavelength \( \lambda \) interacts with one and only one acoustic wave frequency \( f \) satisfying (19.2-1). The frequency of reflected wave is \( \nu+f \).

Although the acoustic wave is modulated, the reflected optical wave is not. That is, under this idealized condition the bandwidth of the modulation is zero!
[ Modulation with a bandwidth B ]
The sound wave is still planar, but the incident light is a beam of width $D$, and then angular divergence $\delta \theta = \lambda / D$.

\[
\delta \theta \approx \frac{(2\pi / v_s) B}{2\pi / \lambda} = \frac{\lambda}{v_s} B. \tag{19.2-2}
\]

**Bandwidth:** $B = \frac{v_s \delta \theta}{\lambda} = \frac{v_s}{D}$ or $B = \frac{1}{T}$, $T = \frac{D}{v_s}$: Transit time of sound wave

Necessary time for the sound wave to interact at all points

The light beam should be more tightly focused in order to increase the bandwidth.
B. Scanners

Relation between the angle of deflection and the sound frequency:

\[ 2\theta \approx \frac{\lambda}{v_s} f, \quad (19.2-5) \]

By changing sound frequency, the deflection angle can be varied. But both the angle of incidence and the sound frequency must be changed simultaneously.

\( \Rightarrow \) Tilting the sound beam!

The angle of tilt must be synchronized with the acoustic frequency \( f \sim \)
[Another method instead of tilting]

To use a sound beam with an angular divergence equal to or greater than the entire range of directions to be scanned. As the sound wave frequency is changed, the Bragg angle is altered and incoming light wave selects only one acoustic plane-wave with the matching direction. ➔ Efficiency is low~

**Figure 19.2-6** Scanning an optical wave by varying the frequency of a sound beam of angular divergence $\delta \theta_s$ over the frequency range $f_0 \leq f \leq f_0 + B$. 
Scan angle
The incident light wave interacts with the sound wave of frequency $f$ at an angle
$\theta = (\lambda / 2v_s) f$, and is deflected by an angle $2\theta = (\lambda / v_s) f$.
Scan angle of deflection to cover the bandwidth $B$ is

$$\Delta \theta = \frac{\lambda}{v_s} B$$

(19.2-6) Scan Angle

Larger scan angles are obtained by use of materials in which the sound velocity is smaller.

The divergence of sound beam should be equal or greater than this angle, that is

$$\delta \theta_s = \Lambda / D_s \geq \Delta \theta$$
**Number of resolvable spots**

Number of resolvable spots:
Number of nonoverlapping angular widths of light within the scanning range.

Angular width of the optical wave: \( \delta \theta = \frac{\lambda}{D} \)
Assuming that \( \delta \theta \ll \delta \theta_s \), the number of resolvable spots is

\[
N = \frac{\Delta \theta}{\delta \theta} = \frac{(\lambda / \nu_s)B}{\lambda / D} = \frac{D}{\nu_s}B \quad (19.2-7) \quad \text{or} \quad N = TB \quad (19.2-8)
\]

*Figure 19.2-7*  Resolvable spots of an acousto-optic scanner.
The acousto-optics scanner as a spectrum analyzer

Sound waves with different frequencies are deflected in different directions. By this, the spectrum of sound wave can be analyzed. ➜ Spectrum analyzer
C. Space switches

Space switch: transfers information carried by one or more optical beams to one or more selected directions.

- Routing an optical beam to one of \( N \) directions by applying acoustic wave of frequencies \( f_i \).

- Routing an optical beam to \( M \) directions simultaneously by applying acoustic wave of frequencies \( f_i \) simultaneously.
Routing each of two \( (N) \) optical beams to a set of specified directions simultaneously by applying acoustic wave of frequencies \( f_i \) successively.

Each pulse duration: \( T/N \)
where, \( T = W/v_s \) : transit time

Spatial light modulator (SLM)

Acoustic wave frequency \( f \) is the same, but with different amplitudes in different segments.
Random access switch that routing each of \( L \) optical beams to \( M \) directions simultaneously by applying acoustic wave of frequencies \( f_i \) successively.

Acoustic cell is divided into \( L \) segments, each of which carries an acoustic wave with \( M \) frequencies.

**Interconnection capacity**

If an acoustic cell is used to route each of \( L \) incoming optical beams to a maximum of \( M \) directions simultaneously, then product \( ML \) cannot exceed the time-bandwidth product \( N=TB \).

\[
ML \leq N \quad (19.2-9)
\]

Interconnection Capacity
Figure 19.2-14 Several examples of dividing the time–bandwidth region $TB$ in the time–frequency diagram into $N = TB$ subdivisions (in this diagram $N = 20$). (a) A scanner: a single time segment containing $N$ frequency segments. (b) A spatial light modulator: $N$ time segments each containing one frequency component. (c) An interconnection switch: $L$ time segments each containing $M = N/L$ frequency segments (in this diagram, $N = 20$, $M = 4$, and $L = 5$).
D. Filters, frequency shifters, and isolators

Tunable acousto-optic filters
Bragg condition, \( \sin \theta = \lambda / 2 \Lambda \). Optical wavelength \( \lambda \) is the function of \( \theta \) and \( \Lambda(f) \). Therefore, an optical wave with a wavelength \( \lambda \) can be selected successively by adjusting the angle \( \theta \) (or the sound frequency \( f \)).

Frequency shifters
Bragg diffracted light is frequency shifted (up or down) by the frequency of sound, \((\nu \pm f)\).
Applications: Optical heterodyning (mixing two optical beams with different frequencies to detect the beat signal), optical FM modulators, and laser Doppler velocimeters, ...

Optical isolators

The frequency of returning light differs from that of original light by twice the sound frequency. A filter may be used to block the returning light. Even without a filter, laser action may be insensitive to the frequency-shifted light.
19. 3 Acousto-optics of anisotropic media

Acoustic waves in anisotropic materials

Position of molecules:
\[ x = (x_1, x_2, x_3) \]

Displacement:
\[ \mathbf{u} = (u_1, u_2, u_3) \]

Strain:
\[ s_{ij} = \frac{1}{2}(\partial u_i / \partial x_j + \partial u_j / \partial x_i) \]

- **Tensile** (Normal) strain \((i=j)\): Strain along the stress
- **Shear** strain \((i\neq j)\): Strain orthogonal to the stress
The photoelastic effect  (Refer to Appendix I~)

Photoelastic effect: Change in refractive index of material by mechanical strain

In the presence of strain, the electric impermeability tensor is modified. Each of the nine functions $\eta_{ij}(s_{kl})$ may be expanded in terms of nine variables $s_{kl}$ in Taylor series.

$$\eta_{ij}(s_{kl}) \approx \eta_{ij}(0) + \sum_{kl} p_{ijkl} s_{kl}, \quad i, j, l, k = 1, 2, 3,$$

(19.3-3)

where, $p_{ijkl} = \partial \eta_{ij} / \partial s_{kl}$ : strain-optic (elasto-optic) tensor

Since both $\{\eta_{ij}\}$ and $\{s_{kl}\}$ are symmetrical tensors, the coefficients $\{p_{ijkl}\}$ are invariant to permutation of $i$ and $j$, and to permutation of $k$ and $l$. There are therefore only six independent values for the set $(i,j) \rightarrow I$ and six independent values for $(k,l) \rightarrow K$. The fourth-rank tensor $\{p_{ijkl}\}$ is thus described by a $6 \times 6$ matrix $p_{IK}$.

Symmetrical tensor (example): $\begin{pmatrix}
\eta_{11} & \eta_{12} & \eta_{13} \\
\eta_{21} = \eta_{12} & \eta_{22} & \eta_{23} \\
\eta_{31} = \eta_{13} & \eta_{32} = \eta_{23} & \eta_{33}
\end{pmatrix}$
Example) Strain-optic matrix for cubic crystal (or isotropic media):

\[
p_{IK} = \begin{bmatrix}
  p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\
  p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\
  p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\
  0 & 0 & 0 & p_{44} & 0 & 0 \\
  0 & 0 & 0 & 0 & p_{44} & 0 \\
  0 & 0 & 0 & 0 & 0 & p_{44}
\end{bmatrix}.
\]

(19.3-4) Strain-Optic Matrix (Cubic Crystal)

Additional constraint: \( p_{44} = \frac{1}{2} (p_{11} + p_{12}) \) \( \Rightarrow \) There are only two independent coefficients!
EXAMPLE 19.3-3. **Longitudinal Acoustic Wave in a Cubic Crystal.** The longitudinal acoustic wave described in Example 19.3-1 travels along one of the axes of a cubic crystal of refractive index \( n \). By substitution of (19.3-1) and (19.3-4) into (19.3-3) we find that the associated strain results in an impermeability tensor with elements,

\[
\begin{align*}
\eta_{11} &= \eta_{22} = \frac{1}{n^2} + p_{12}S_0 \cos(\Omega t - qz) \\
\eta_{33} &= \frac{1}{n^2} + p_{11}S_0 \cos(\Omega t - qz) \\
\eta_{ij} &= 0, \quad i \neq j.
\end{align*}
\]  

(19.3-5)  
(19.3-6)  
(19.3-7)

Thus, the initially optically isotropic cubic crystal becomes a uniaxial crystal with the optic axis in the direction of the acoustic wave (z direction) and with ordinary and extraordinary refractive indexes, \( n_o \) and \( n_e \), given by

\[
\frac{1}{n_o^2} = \frac{1}{n_e^2} + p_{12}S_0 \cos(\Omega t - qz)
\]

(19.3-8)

From example 19.3-1,

\[
\sum p_{ik}s_{ij} = \begin{pmatrix}
p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\
p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\
p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & p_{44} & 0 & 0 \\
0 & 0 & 0 & p_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & p_{44}
\end{pmatrix} \begin{pmatrix}
0 \\
0 \\
0 \\
S_0 \cos(\Omega t - qz) \\
0 \\
0
\end{pmatrix} = \begin{pmatrix}
p_{12}S_0 \cos(\Omega t - qz) \\
p_{12}S_0 \cos(\Omega t - qz) \\
p_{11}S_0 \cos(\Omega t - qz) \\
0 \\
0 \\
0
\end{pmatrix}
\]
\[
\frac{1}{n_e^2} = \frac{1}{n^2} + p_{11} S_0 \cos(\Omega t - qz). \tag{19.3-9}
\]

The shape of the index ellipsoid is altered periodically in time and space in the form of a wave, but the principal axes remain unchanged (see Fig. 19.3-2). Since the change of the refractive indexes is usually small, the second terms in (19.3-8) and (19.3-9) are small, so that the approximation \((1 + \Delta)^{-1/2} \approx 1 - \Delta/2\), when \(|\Delta| \ll 1\), may be applied to approximate (19.3-8) and (19.3-9) by

\[
n_o \approx n - \frac{1}{2} n^3 p_{12} S_0 \cos(\Omega t - qz) \tag{19.3-10}
\]

\[
n_e \approx n - \frac{1}{2} n^3 p_{11} S_0 \cos(\Omega t - qz). \tag{19.3-11}
\]

**Figure 19.3-2** A longitudinal acoustic wave traveling in the \(z\) direction in a cubic crystal alters the shape of the index ellipsoid from a sphere into an ellipsoid of revolution with dimensions varying sinusoidally with time and an axis in the \(z\) direction.
Bragg diffraction (Refer to Appendix II~)

Bragg condition: \( k_r = k + q \).  

\[
k = \left(\frac{2\pi}{\lambda_o}\right)n, \quad k_r = \left(\frac{2\pi}{\lambda_o}\right)n_r, \quad \text{and} \quad q = \left(\frac{2\pi}{\Lambda}\right)
\]

\[
\frac{2\pi}{\lambda_o} n_r \cos \theta_r = \frac{2\pi}{\lambda_o} n \cos \theta \quad (z\text{-component}) \quad (19.3-16)
\]

\[
\frac{2\pi}{\lambda_o} n_r \sin \theta_r + \frac{2\pi}{\lambda_o} n \sin \theta = \frac{2\pi}{\Lambda}, \quad (x\text{-component}) \quad (19.3-17)
\]

\[
n_r \cos \theta_r = n \cos \theta \quad (19.3-18a)
\]

\[
n_r \sin \theta_r + n \sin \theta = \frac{\lambda_o}{\Lambda}. \quad (19.3-18b)
\]
From (19.3-18), when $\theta_r = \pi/2$ \Rightarrow $0 = n\cos\theta \Rightarrow \theta = \pm\pi/2$

\[ n_r + nsin\theta = \frac{\lambda_0}{\Lambda} \]

1) $\theta = +\pi/2 \Rightarrow n_r + n = \frac{\lambda_0}{\Lambda}$
2) $\theta = -\pi/2 \Rightarrow n_r - n = \frac{\lambda_0}{\Lambda}$

\[ n_r \pm n = \frac{\lambda_0}{\Lambda} \] (19.3-19)

Figure 19.3-4 Wavevector diagram for reflection of an optical wave from an acoustic wave.
Appendix

I. Photoelastic effect

II. Bragg diffraction (Coupled-wave theory)

III. Raman-Nath diffraction

References:
I. Photoelastic effect

Photoelastic effect

: Mechanical strain \(\rightarrow\) Index of refraction

Index ellipsoid for Principal axes:

\[
\frac{x^2}{n_1^2} + \frac{y^2}{n_2^2} + \frac{z^2}{n_3^2} = 1
\]

Strain tensor elements, \(S\):

\[
S_{xx} = \frac{\partial u}{\partial x}, \quad S_{yy} = \frac{\partial v}{\partial y}, \quad S_{zz} = \frac{\partial w}{\partial z}
\]

\[
S_{xy} = \left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right) = S_{yx}
\]

\[
S_{yz} = \left(\frac{\partial v}{\partial z}\right) + \left(\frac{\partial w}{\partial y}\right) = S_{zy}
\]

\[
S_{xz} = \left(\frac{\partial u}{\partial z}\right) + \left(\frac{\partial w}{\partial x}\right) = S_{zx}
\]

\(
S_1 \equiv S_{xx}, \quad S_2 \equiv S_{yy}, \quad S_3 \equiv S_{zz} : \text{Tensile(Normal) strain}
\)

\(
S_4 \equiv S_{yz}, \quad S_5 \equiv S_{zx}, \quad S_6 \equiv S_{xy} : \text{Shear strain}
\)

where, \(u, v, w\) : displacements along the \(x, y, z\) axes
Change in index of refraction due to the mechanical strain:

$$\Delta \left( \frac{1}{n^2} \right)_i = \sum_{j=1}^{6} p_{ij} S_j (i=1,\ldots,6)$$

where, $P_{ij}$ : Elasto-Optic (Strain Optic) Coefficient (6x6 matrix) ➔ Table 9.1 / 9.2

The equation of the index ellipsoid in the presence of a strain field:

$$x^2 \left( \frac{1}{n_1^2} + \sum_j p_{1j} S_j \right) + y^2 \left( \frac{1}{n_2^2} + \sum_j p_{2j} S_j \right) + z^2 \left( \frac{1}{n_3^2} + \sum_j p_{3j} S_j \right) + 2yz \sum_j p_{4j} S_j + 2xz \sum_j p_{5j} S_j + 2xy \sum_j p_{6j} S_j = 1$$

Yariv, P. Yeh, “Optical Waves in Crystals”, John Wiley & Sons
### Table 9.1. Forms of the Elasto-optic Coefficients in Contracted Notation for all Classes of Crystal Symmetry

<table>
<thead>
<tr>
<th>Triclinic (36)†</th>
<th>Trigonal (8) classes 3m, 32, 3m</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{11}$ $p_{12}$ $p_{13}$ $p_{14}$ $p_{15}$ $p_{16}$</td>
<td>$p_{11}$ $p_{12}$ $p_{13}$ $p_{14}$ 0 0</td>
</tr>
<tr>
<td>$p_{21}$ $p_{22}$ $p_{23}$ $p_{24}$ $p_{25}$ $p_{26}$</td>
<td>$p_{12}$ $p_{11}$ $p_{13}$ $-p_{14}$ 0 0</td>
</tr>
<tr>
<td>$p_{31}$ $p_{32}$ $p_{33}$ $p_{34}$ $p_{35}$ $p_{36}$</td>
<td>$p_{13}$ $p_{13}$ $p_{13}$ $0$ 0 0</td>
</tr>
<tr>
<td>$p_{41}$ $p_{42}$ $p_{43}$ $p_{44}$ $p_{45}$ $p_{46}$</td>
<td>$p_{41}$ $-p_{41}$ 0 $p_{44}$ 0 0</td>
</tr>
<tr>
<td>$p_{51}$ $p_{52}$ $p_{53}$ $p_{54}$ $p_{55}$ $p_{56}$</td>
<td>0 0 0 0 $p_{44}$ $p_{41}$</td>
</tr>
<tr>
<td>$p_{61}$ $p_{62}$ $p_{63}$ $p_{64}$ $p_{65}$ $p_{66}$</td>
<td>0 0 0 0 $p_{14} \frac{1}{2}(p_{11} - p_{12})$</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Monoclinic (20)</th>
<th>Orthorhombic (12)</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>$p_{21}$ $p_{22}$ $p_{23}$ 0 0 $p_{25}$</td>
<td>$p_{21}$ $p_{22}$ $p_{23}$ 0 0 0</td>
</tr>
<tr>
<td>$p_{31}$ $p_{32}$ $p_{33}$ 0 0 $p_{35}$</td>
<td>$p_{31}$ $p_{32}$ $p_{33}$ 0 0 0</td>
</tr>
<tr>
<td>0 0 0 $p_{44}$ 0 $p_{46}$</td>
<td>0 0 0 0 $p_{44}$ 0</td>
</tr>
<tr>
<td>$p_{51}$ $p_{52}$ $p_{53}$ 0 0 $p_{55}$</td>
<td>0 0 0 0 $p_{55}$ 0</td>
</tr>
<tr>
<td>0 0 0 $p_{64}$ 0 $p_{66}$</td>
<td>0 0 0 0 0 $p_{66}$</td>
</tr>
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</table>

<table>
<thead>
<tr>
<th>Tetragonal (10) classes 4, 4, 4/m</th>
<th>Tetragonal (7) classes 4mm, 422, 4/mmm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{11}$ $p_{12}$ $p_{13}$ 0 0 0</td>
<td>$p_{11}$ $p_{12}$ $p_{13}$ 0 0 0</td>
</tr>
<tr>
<td>$p_{12}$ $p_{11}$ $p_{13}$ 0 0 $-p_{16}$</td>
<td>$p_{12}$ $p_{11}$ $p_{13}$ 0 0 0</td>
</tr>
<tr>
<td>$p_{31}$ $p_{32}$ $p_{33}$ 0 0 0</td>
<td>$p_{31}$ $p_{31}$ $p_{33}$ 0 0 0</td>
</tr>
<tr>
<td>0 0 0 $p_{44}$ $p_{45}$ 0</td>
<td>0 0 0 0 $p_{44}$ 0</td>
</tr>
<tr>
<td>0 0 0 $-p_{45}$ $p_{44}$ 0</td>
<td>0 0 0 0 $p_{14} \frac{1}{2}(p_{11} - p_{12})$</td>
</tr>
<tr>
<td>$p_{61}$ $-p_{61}$ 0 0 0 $p_{66}$</td>
<td>0 0 0 0 0 $p_{66}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Trigonal (12) classes 3, 3</th>
<th>Cubic (4) classes 23, m3</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>$p_{12}$ $p_{11}$ $p_{13}$ $-p_{14}$ $-p_{15}$ $-p_{16}$</td>
<td>$p_{21}$ $p_{11}$ $p_{12}$ 0 0 0</td>
</tr>
<tr>
<td>$p_{31}$ $p_{31}$ $p_{33}$ 0 0 0</td>
<td>$p_{12}$ $p_{21}$ $p_{11}$ 0 0 0</td>
</tr>
<tr>
<td>$p_{41}$ $-p_{41}$ 0 $p_{44}$ $p_{45}$ $-p_{51}$</td>
<td>0 0 0 $p_{44}$ 0 0</td>
</tr>
<tr>
<td>$p_{51}$ $-p_{51}$ 0 $-p_{45}$ $p_{44}$ $p_{41}$</td>
<td>0 0 0 0 $p_{44}$ 0</td>
</tr>
<tr>
<td>$-p_{16}$ $p_{16}$ 0 $-p_{15}$ $p_{14}$ $\frac{1}{2}(p_{11} - p_{12})$</td>
<td>0 0 0 0 0 $p_{44}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hexagonal (8) classes 6, 6, 6/m</th>
<th>Hexagonal (6) classes 6m2, 6mm, 622, 6/mmm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{11}$ $p_{12}$ $p_{13}$ 0 0 $p_{16}$</td>
<td>$p_{11}$ $p_{12}$ $p_{13}$ 0 0 0</td>
</tr>
<tr>
<td>$p_{12}$ $p_{11}$ $p_{13}$ 0 0 $-p_{16}$</td>
<td>$p_{12}$ $p_{11}$ $p_{13}$ 0 0 0</td>
</tr>
<tr>
<td>$p_{31}$ $p_{31}$ $p_{33}$ 0 0 0</td>
<td>$p_{31}$ $p_{31}$ $p_{33}$ 0 0 0</td>
</tr>
<tr>
<td>0 0 0 $p_{44}$ $p_{45}$ 0</td>
<td>0 0 0 0 $p_{45}$ 0</td>
</tr>
<tr>
<td>0 0 0 $-p_{45}$ $p_{44}$ 0</td>
<td>0 0 0 0 $-p_{45} \frac{1}{2}(p_{11} - p_{12})$</td>
</tr>
<tr>
<td>$p_{61}$ $-p_{61}$ 0 0 0 $p_{66}$</td>
<td>0 0 0 0 0 $p_{66}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cubic (3) classes 43m, 432, m3m</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
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<td>$p_{11}$ $p_{12}$ $p_{13}$ 0 0 0</td>
</tr>
<tr>
<td>$p_{21}$ $p_{11}$ $p_{12}$ 0 0 0</td>
<td>$p_{21}$ $p_{11}$ $p_{12}$ 0 0 0</td>
</tr>
<tr>
<td>$p_{12}$ $p_{21}$ $p_{11}$ 0 0 0</td>
<td>$p_{12}$ $p_{12}$ $p_{11}$ 0 0 0</td>
</tr>
<tr>
<td>0 0 0 $p_{44}$ 0 0</td>
<td>0 0 0 0 $p_{44}$ 0</td>
</tr>
<tr>
<td>0 0 0 0 $p_{44}$ 0</td>
<td>0 0 0 0 $p_{44}$ 0</td>
</tr>
<tr>
<td>0 0 0 0 0 $p_{44}$</td>
<td>0 0 0 0 0 $p_{44}$</td>
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Table 9.1.  (Continued).

<table>
<thead>
<tr>
<th>Isotropic (2)</th>
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<tbody>
<tr>
<td>$P_{11}$   $P_{12}$ $P_{12}$   0     0     0</td>
</tr>
<tr>
<td>$P_{12}$   $P_{11}$ $P_{12}$   0     0     0</td>
</tr>
<tr>
<td>$P_{12}$   $P_{12}$ $P_{11}$   0     0     0</td>
</tr>
<tr>
<td>0     0     0     $\frac{1}{2}(p_{11} - p_{12})$ 0     0</td>
</tr>
<tr>
<td>0     0     0     0     $\frac{1}{2}(p_{11} - p_{12})$ 0</td>
</tr>
<tr>
<td>0     0     0     0     0     $\frac{1}{2}(p_{11} - p_{12})$</td>
</tr>
</tbody>
</table>

1 The number inside the parentheses indicates the number of independent coefficients.
Example) Sound wave propagating along the z direction in water

Sound wave: \( w(z, t) = A \hat{z} \cos(\Omega t - Kz) \)

\[
S_1 = S_{xx} = \partial u / \partial x = 0, \quad S_2 = S_{yy} = \partial v / \partial y = 0, \quad S_3 = S_{zz} = \partial w / \partial z = K \sin(\Omega t - Kz) = S \sin(\Omega t - Kz)
\]

\[
S_4 = S_{yz} = (\partial v / \partial z) + (\partial w / \partial y) = 0, \quad S_5 = S_{zx} = (\partial u / \partial z) + (\partial w / \partial x) = 0, \quad S_6 = S_{xy} = (\partial u / \partial y) + (\partial v / \partial x) = 0
\]

Elasto-Optic Coefficient for the water (isotropic, Table 9.1):

\[
p_{ij} = \begin{bmatrix}
    p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\
p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\
p_{12} & p_{12} & p_{11} & 0 & 0 & 0 \\
    0 & 0 & 0 & 1/2(p_{11} - p_{12}) & 0 & 0 \\
    0 & 0 & 0 & 0 & 1/2(p_{11} - p_{12}) & 0 \\
    0 & 0 & 0 & 0 & 0 & 1/2(p_{11} - p_{12})
\end{bmatrix}
\]

\[
\Rightarrow \Delta (\frac{1}{n^2})_1 = \Delta (\frac{1}{n^2})_2 = p_{12} S \sin(\Omega t - Kz),
\]

\[
\Delta (\frac{1}{n^2})_3 = p_{11} S \sin(\Omega t - Kz),
\]

\[
\Delta (\frac{1}{n^2})_{4,5,6} = 0
\]

The new index ellipsoid:

\[
x^2 \left[ \frac{1}{n^2} + p_{12} S \sin(\Omega t - Kz) \right] + y^2 \left[ \frac{1}{n^2} + p_{12} S \sin(\Omega t - Kz) \right] + z^2 \left[ \frac{1}{n^2} + p_{11} S \sin(\Omega t - Kz) \right] = 1
\]
Example) y-polarized Shear wave propagating along the z direction in Ge

Sound wave: \( v(z, t) = A \hat{y} \cos(\Omega t - Kz) \)

\[ S_1 \equiv S_{xx} = \partial u/\partial x = 0, \quad S_2 \equiv S_{yy} = \partial v/\partial y = 0, \quad S_3 \equiv S_{zz} = \partial w/\partial z = 0, \]

\[ S_4 \equiv S_{yz} = (\partial v/\partial z) + (\partial w/\partial y) = K A \sin(\Omega t - Kz) \equiv S \sin(\Omega t - Kz), \]

\[ S_5 \equiv S_{zx} = (\partial u/\partial z) + (\partial w/\partial x) = 0, \quad S_6 \equiv S_{xy} = (\partial u/\partial y) + (\partial v/\partial x) = 0 \]

Elasto-Optic Coefficient for the Ge (cubic, Table 9.1):

\[ p_{ij} = \begin{bmatrix} p_{11} & p_{12} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ p_{12} & p_{11} & p_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & p_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & p_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & p_{44} \end{bmatrix} \]

\[ \Rightarrow \Delta \left( \frac{1}{n^2} \right)_1 = \Delta \left( \frac{1}{n^2} \right)_2 = \Delta \left( \frac{1}{n^2} \right)_3 = 0, \]

\[ \Delta \left( \frac{1}{n^2} \right)_4 = p_{44} S \sin(\Omega t - Kz), \]

\[ \Delta \left( \frac{1}{n^2} \right)_{5,6} = 0 \]

The new index ellipsoid:

\[ \frac{1}{n^2} (x^2 + y^2 + z^2) + 2yzp_{44} S \sin(\Omega t - Kz) = 1 \]
II. Bragg diffraction (Coupled-wave theory)

In this regime, we can no longer consider the refractive index perturbation to act as a thin phase grating. We should consider the propagation equation of light:

$$\nabla^2 e = \mu_0 \frac{\partial^2}{\partial t^2} [\varepsilon + \Delta \varepsilon \sin(\Omega t - Kz)] e$$

$$\Omega \ll \omega$$

$$\Rightarrow (\nabla^2 - \varepsilon \mu_0 \frac{\partial^2}{\partial t^2}) e \approx \mu_0 \Delta \varepsilon \sin(\Omega t - Kz) \frac{\partial^2 e}{\partial t^2}$$

Converting to 2-D problem (no y dependence):

$$\Rightarrow \frac{\partial^2 e}{\partial x^2} + \frac{\partial^2 e}{\partial z^2} - \varepsilon \mu_0 \frac{\partial^2 e}{\partial t^2} = \frac{1}{2i} \mu_0 \Delta \varepsilon (e^{i(\Omega t - Kz)} - e^{-i(\Omega t - Kz)}) \frac{\partial^2 e}{\partial t^2}$$

Total e-field:

$$e = e_0 + e_+ + e_-$$

where,

$$e_0 = A_0(x,z)e^{i(\omega x - kr)} = A_0(x,z)e^{i(\omega x - \alpha x - \beta z)}$$

$$e_\pm = A(x,z)e^{i(\omega_\pm t - k_\pm x)} = A_\pm(x,z)e^{i[\omega_\pm \Omega t - \alpha_\pm x - \beta_\pm z]}$$
Let, \( \omega^2 \varepsilon_0 = k^2 = \alpha^2 + \beta^2 \)

\((\omega \pm \Omega)^2 \varepsilon_0 = k^2 = \alpha^2 + \beta^2\)

And, slow varying approximation;

\[ | \frac{\partial^2 A}{\partial x^2} | \ll | k \frac{\partial A}{\partial x} |, | \frac{\partial^2 A}{\partial z^2} | \ll | k \frac{\partial A}{\partial z} | \]

\[-2i \left( \alpha \frac{\partial A_0}{\partial x} + \beta \frac{\partial A_0}{\partial z} \right) e^{i(\omega x - \alpha x - \beta z)}
-2i \left( \alpha_+ \frac{\partial A_+}{\partial x} + \beta_+ \frac{\partial A_+}{\partial z} \right) e^{i[(\omega + \Omega) t - \alpha_+ x - \beta_+ z]}
-2i \left( \alpha_- \frac{\partial A_-}{\partial x} + \beta_- \frac{\partial A_-}{\partial z} \right) e^{i[(\omega - \Omega) t - \alpha_- x - \beta_- z]}
= -\frac{1}{2i} \mu_0 \Delta \varepsilon \left[ e^{i(\Omega t - Kz)} - e^{-i(\Omega t - Kz)} \right]
\cdot \left[ \omega A_0 e^{i(\omega t - \alpha x - \beta z)} + (\omega + \Omega)^2 A_+ e^{i[(\omega + \Omega) t - \alpha_+ x - \beta_+ z]} + (\omega - \Omega)^2 A_- e^{i[(\omega - \Omega) t - \alpha_- x - \beta_- z]} \right] \]
\[ -2i \left( \alpha \frac{\partial A_0}{\partial x} + \beta \frac{\partial A_0}{\partial z} \right) e^{i(\alpha x + \beta z)} = -\frac{1}{2i} \omega^2 \mu_0 \Delta \varepsilon \left[ A_+ e^{-i[\alpha_x + (\beta_+ - K)z]} + A_- e^{-i[\alpha_x + (\beta_- + K)z]} \right] \\
-2i \left( \alpha_\pm \frac{\partial A_\pm}{\partial x} + \beta_\pm \frac{\partial A_\pm}{\partial z} \right) e^{-i(\alpha_\pm x + \beta_\pm z)} = \mp \frac{1}{2i} \omega^2 \mu_0 \Delta \varepsilon A_0 e^{-i[\alpha x + (\beta \pm K)z]} \]
(1) Small Bragg angle diffraction

\[ \theta \approx 0 \implies \frac{\partial A}{\partial z} \approx 0 \]

\[ -2i\alpha \frac{dA_0}{dx} e^{-i(\alpha x + \beta z)} = -\frac{1}{2i} \omega^2 \mu_0 \Delta \varepsilon \left[ -A_+ e^{-i[\alpha x + (\beta + K)z]} + A_- e^{-i[\alpha x + (\beta - K)z]} \right] \]

\[ -2i\alpha \pm \frac{dA_\pm}{dx} e^{-i(\alpha_x \pm \beta_{\pm} z)} = \pm \frac{1}{2i} \omega^2 \mu_0 \Delta \varepsilon A_0 e^{-i[\alpha x + (\beta \pm K)z]} \]

\[ \Rightarrow \beta_+ = \beta + K \quad \& \quad \beta_- = \beta - K \quad : \text{Bragg condition} \]
These equations have a solution for $A_{\pm}$ when only $\alpha_{\pm} \approx \alpha$.
The solutions for $\alpha_{+} = \alpha$, $\alpha_{-} = \alpha$ are independent each other, so let $\alpha_{+} = \alpha$

\[
\begin{align*}
\frac{d\tilde{A}_0}{dx} &= k\tilde{A}_+ e^{ix\Delta \alpha} \\
\frac{d\tilde{A}_-}{dx} &= -k\tilde{A}_0 e^{-ix\Delta \alpha}
\end{align*}
\Rightarrow \quad \frac{d^2\tilde{A}_0}{dx^2} - i(\Delta \alpha) \frac{d\tilde{A}_0}{dx} + \kappa^2 \tilde{A}_0 = 0
\]

solution:

\[
\tilde{A}_0(x) = C_0 e^{ix(\frac{1}{2}\Delta \alpha + \delta)} + D_0 e^{ix(\frac{1}{2}\Delta \alpha - \delta)}
\]

where, $\delta = \left[k^2 + \frac{1}{4}(\Delta \alpha)^2\right]^{1/2}$

\[
\tilde{A}_+(x) = \left(C_+ e^{ix(\frac{1}{2}\Delta \alpha + \delta)} + D_+ e^{ix(\frac{1}{2}\Delta \alpha - \delta)}\right)e^{-ix\Delta \alpha}
\]

where, $C_+ \equiv \left(i/k\right)\left(\frac{1}{2}\Delta \alpha + \delta\right)C_0$

$D_+ \equiv \left(i/k\right)\left(\frac{1}{2}\Delta \alpha - \delta\right)D_0$
Diffraction efficiency, $\eta$

\[ \tilde{A}_0(x=0)=1, \quad \tilde{A}_+(x=0)=0 \text{ (initial condition)} \]

\[ \Rightarrow \quad C_0=\frac{1}{2}-\left(\frac{1}{4\delta}\right)\Delta \alpha, \quad D_0=\frac{1}{2}+\left(\frac{1}{4\delta}\right)\Delta \alpha \]

0-th and 1-st diffraction powers:

\[ P_0(x)=\left|\tilde{A}_0(x)\right|^2=\cos^2(\delta x)+\left(\Delta \alpha/2\delta\right)^2 \sin^2(\delta x) \]

\[ P_+(x)=\left|\tilde{A}_+(x)\right|^2=\left(\kappa/\delta\right)^2 \sin^2(\delta x) \]

i) \[ P_0(x)+P_+(x)=1\left(\delta^2=\kappa^2+\frac{1}{4}(\Delta \alpha)^2\right) \]

ii) Maximum transfer: \[ \Delta \alpha=\alpha-\alpha_+=0; \quad \alpha=\alpha_+ \]

\[ P_0(x)=\cos^2(\kappa x) \]

\[ P_+(x)=\sin^2(\kappa x) \]

Diffraction efficiency: \[ \eta=P_+(L)=\sin^2(\kappa L) \]

\[ \eta=1\left<L=\pi/2\kappa, 3\pi/2\kappa, \ldots\right> \]
\[ \Delta \bar{\varepsilon} = -\frac{1}{\varepsilon_0} \bar{\varepsilon} \Delta \left( \frac{1}{n^2} \right) \bar{\varepsilon} \varepsilon \quad \Rightarrow \quad \Delta \varepsilon = \varepsilon_0 n^4 \bar{p} \bar{S} \quad : \text{scalar expression} \]

\[ \therefore \ k = \frac{\omega n^3 \bar{p} \bar{S}}{4c \cos \theta_B} \left( \alpha = \alpha_+ = \left( \frac{\omega}{c} \right) n \cos \theta_B \right) \]

Acoustic intensity : \( I_a = \frac{1}{2} \rho v_a^3 \bar{S}^2 \)

\[ \Rightarrow \quad k = \frac{\pi}{\sqrt{2 \lambda \cos \theta_B}} (M_2 I_a)^{1/2} \]

where, \( M_2 = n^6 \bar{p}^2 / \rho v_a^3 \) : Figure of Merit

Diffraction efficiency :

\[ \eta = \sin^2 \left[ \frac{\pi}{\sqrt{2 \lambda \cos \theta_B}} (M_2 I_a)^{1/2} L \right] \]
Acoustic intensity for maximum efficiency:

\[ I_{am} = \frac{\lambda^2 \cos^2 \theta_B}{2M_2 L^2} \]

Acoustic power for maximum efficiency (LH cross-section, maximum impedance matching case):

\[ P_{am} = I_{a LH} = \frac{\lambda^2 \cos^2 \theta_B \left( \frac{H}{L} \right)}{2M_2} \propto \frac{1}{M_2} \]

Diffraction figure of merit of the material relative to water
(2) Large Bragg angle diffraction

\[ \theta \approx \pi/2 \Rightarrow \frac{\partial A}{\partial x} \approx 0 \]

\[ \beta \frac{\partial A_0}{\partial z} e^{-i(\alpha x + \beta z)} = \frac{1}{4} \omega^2 \mu_0 \Delta \varepsilon [A_+ e^{-i(\alpha x + (\beta + K) z)} - A_- e^{-i(\alpha x + (\beta - K) z)}] \]

\[ \beta_{\pm} \frac{\partial A_{\pm}}{\partial z} e^{-i(\alpha_{\pm} x + \beta_{\pm} z)} = \mp \frac{1}{4} \omega^2 \mu_0 \Delta \varepsilon A_0 e^{-i(\alpha x + (\beta_{\pm} K) z)} \]

- x-dependent term \( \Rightarrow \alpha_{\pm} = \alpha \)
These equations have a solution for $A_\pm$ when only $\beta_\pm \approx \beta \pm K$.
The two solutions are independent each other, so let

1) $\beta_+ = \beta + K$ (Co-directional coupling)

$$\frac{\partial \tilde{A}_0}{\partial z} = \frac{\beta}{|\beta|} \sigma \tilde{A}_0 e^{i(\Delta \beta)z}$$

$$\frac{\partial \tilde{A}_+}{\partial z} = -\frac{\beta_+}{|\beta_+|} \sigma \tilde{A}_0 e^{-i(\Delta \beta)z}$$

where, $\sigma = \frac{\omega^2 \mu_0 \Delta \varepsilon}{4(|\beta||\beta_+|)^{1/2}} = \frac{\omega^2 \Delta \varepsilon}{4c^2 \varepsilon_0 |\beta||\beta_+|^{1/2}}$

$\tilde{A}_0 = \left(\frac{|\beta|}{2\omega \mu_0}\right)^{1/2} A_0$

$\tilde{A}_+ = \left(\frac{|\beta_+|}{2\omega \mu_0}\right)^{1/2} A_+$

$\Delta \beta = \beta - \beta_+ + k$

Solutions:

$$\tilde{A}_0(z) = C_0 e^{iz(\frac{1}{2} \Delta \beta + \delta)} + D_0 e^{iz(\frac{1}{2} \Delta \beta - \delta)}$$

where, $\delta = \left[\delta^2 + \frac{1}{4}(\Delta \beta)^2\right]^{1/2}$

$$\tilde{A}_+(z) = C_+ e^{iz(\frac{1}{2} \Delta \beta + \delta)} + D_+ e^{iz(\frac{1}{2} \Delta \beta - \delta)} e^{-iz\Delta \beta}$$

where, $C_+ = \left(\frac{i}{\sigma}\right)\left(\frac{1}{2} \Delta \beta + \delta\right) C_0$

$D_+ = \left(\frac{i}{\sigma}\right)\left(\frac{1}{2} \Delta \beta - \delta\right) D_0$
Diffraction efficiency, $\eta$

\[ \tilde{A}_0(z = 0) = 1, \quad \tilde{A}_+(z = 0) = 0 \]

\[ \Rightarrow \quad C_0 = \frac{1}{2} - \left( \frac{1}{4\delta} \right) \Delta \beta, \quad D_0 = \frac{1}{2} + \left( \frac{1}{4\delta} \right) \Delta \beta \]

0-th and 1-st diffraction powers:

\[ P_0(z) = \left| \tilde{A}_0(z) \right|^2 = \cos^2(\delta \kappa) + \left( \frac{\Delta \beta}{2\delta} \right)^2 \sin^2(\delta \kappa) \]

\[ P_+(x) = \left| \tilde{A}_+(z) \right|^2 = \left( \frac{\sigma}{\delta} \right)^2 \sin^2(\delta \kappa) \]

Diffraction efficiency:

\[ \eta = \left( \frac{\sigma}{\delta} \right)^2 \sin^2(\delta L) \]

\[ = \frac{1}{1 + \left( \frac{\Delta \beta}{2\sigma \delta} \right)^2} \sin^2 \left[ \sigma L \left\{ 1 + \frac{(\Delta \beta)^2}{4\sigma^2} \right\}^{1/2} \right] \]
2) $\beta_-=\beta-K$ (Counter-directional coupling)

\[
\frac{\partial \tilde{A}_0}{\partial z} = -\sigma \tilde{A}_- e^{i(\Delta \beta)z}
\]

\[
\frac{\partial \tilde{A}_-}{\partial z} = -\sigma \tilde{A}_0 e^{-i(\Delta \beta)z}
\]

where, $\Delta \beta = \beta - \beta_- - K$

Solutions:

\[
\tilde{A}_0(z) = e^{i(\Delta \beta)z} \left( P_1 e^{gz} + Q_1 e^{-gz} \right) + D_0 e^{iz(\frac{1}{2} \Delta \beta - \delta)}
\]

\[
\tilde{A}_-(z) = -\left( \frac{1}{\sigma} \right) e^{-i(\Delta \beta)z/2} \left[ P_1 \left( g + i \frac{1}{2} \Delta \beta \right) e^{gz} - Q_1 \left( g - i \frac{1}{2} \Delta \beta \right) e^{-gz} \right]
\]

where, $g = \left( \sigma^2 - \frac{1}{4} (\Delta \beta)^2 \right)^{\frac{1}{2}}$
\[ \tilde{A}_0(z = 0) = 1, \quad \tilde{A}_-(z = D) = 0 \]

\[ \Rightarrow P_0(D) = |\tilde{A}_0|^2 = \frac{g^2}{g^2 \cosh^2 gD + \left(\frac{1}{2} \Delta \beta\right)^2 \sinh^2 gD} \]

\[ P_-(D) = |\tilde{A}_-|^2 = \frac{\left\{g^2 + \left(\frac{1}{2} \Delta \beta\right)^2\right\} \sinh^2 gD}{g^2 \cosh^2 gD + \left(\frac{1}{2} \Delta \beta\right)^2 \sinh^2 gD} \]

# Application: DBR reflector
### III. Raman-Nath diffraction

**Acousto-Optic effect**

Bragg diffraction & Raman-Nath diffraction

- **Bragg diffraction**:
  - acoustic wave vector is well defined

- **Raman-Nath diffraction**:
  - acoustic wave vector has an angular distribution

**Vector Representation**

- **Raman-Nath diffraction**
  - acoustic wave vector

- **Bragg diffraction**
  - acoustic wave vector is well defined

### Equations

- **Spread angle of Acoustic wave**:
  \[ \Phi \sim \frac{\Lambda}{L} \]

- **Diffraction angle of Light**:
  \[ \theta_B \sim \frac{\lambda}{2n\Lambda} \]

- **Dimensionless parameter**:
  \[ Q = \frac{4\pi\theta_B}{\Phi} = \frac{2\pi\lambda L}{n\Lambda^2} \]

- \( Q > 1 \) : Bragg diffraction
- \( Q < 1 \) : Raman-Nath diffraction

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**Nonlinear Optics Lab. Hanyang Univ.**
Example) Water, $n=1.33$, $\mathcal{W}=6\text{MHz}$ ($v_s=1,500\text{ m/s}$), $l=632.8\text{ nm}$

$$\Lambda = \frac{v_s}{\Omega} = 250\mu\text{m}$$

$L \ll \Lambda^2 n/(2\pi\lambda) \approx 2\text{ cm}$ : Raman-Nath Regime

$>> 2\text{ cm}$ : Bragg Regime

Bragg diffraction : Single order diffraction

Raman-Nath diffraction : Multiple order diffraction
Moving periodic refractive index grating: \( n(z, t) = n_0 + \Delta n \sin(\Omega t - Kz) \)

Consider \( L \) is small enough so that the medium behave as a thin phase grating,

\[
\Delta \phi = \frac{2\pi}{\lambda} n(z, t)L = \phi_0 + \phi_1 \sin(\Omega t - Kz)
\]

where, \( \phi_1 = (2\pi/\lambda)n_0L, \phi_1 = (2\pi/\lambda)\Delta nL \)

The transmitted field on the plane \( x=L \):

\[
E_t = E_0 e^{i[\omega t - \phi_0 - \phi_1 \sin(\Omega t - Kz)]}
\]

\[
e^{-i\xi \sin \theta} = \sum_{m=-\infty}^{\infty} J_m(\xi)e^{-im\theta}
\]

\[
E_t = E_0 e^{i(\omega t - \phi_0)} \left[ J_0(\phi_1) - J_1(\phi_1)\{e^{i\theta} - e^{-i\theta}\} + J_2(\phi_1)\{e^{2i\theta} - e^{-2i\theta}\} + \cdots \right]
\]

where, \( \theta \equiv (\Omega t - Kz) \)
\[ E_t = E_0 J_0 (\phi_1) e^{i(\omega t - \phi_0)} \]

\[ - E_0 J_1 (\phi_1) [e^{-i(\omega + \Omega)t - Kz_{-} - \phi_0} - e^{-i(\omega - \Omega)t + Kz_{+} - \phi_0}] \]

\[ + E_0 J_2 (\phi_1) [e^{-i(\omega + 2\Omega)t - 2Kz_{-} - \phi_0} - e^{-i(\omega - 2\Omega)t + 2Kz_{+} - \phi_0}] \]

\[ - \ldots \]

\[ E_t^0 = E_0 J_0 (\phi_1) e^{i(\omega t - k(x-L) - \phi_0)} \]

- **Amplitude reduction**

- **Frequency**: \( \omega + \Omega, \quad \omega - \Omega \)

- **Wave vector**: \( k^+ = \left( (\omega + \Omega)^2 / c^2 - K^2 \right)^{\frac{1}{2}} \),
  \( k^- = \left( (\omega - \Omega)^2 / c^2 - K^2 \right)^{\frac{1}{2}} \)

- **Propagation in** \( x > L \):
  \[- E_0 J_1 (\phi_1) e^{i(\omega + \Omega)t - k^+(x-L) - Kz_{-} - \phi_0} : +1 \text{ order} \]
  \[ E_0 J_1 (\phi_1) e^{i(\omega - \Omega)t - k^- (x-L) + Kz_{-} - \phi_0} : -1 \text{ order} \]

- **Diffraction angles**:
  \[ \sin \theta_1 = \frac{K}{k} = \frac{\lambda}{n\Lambda} \]
  \[ \sin \theta_{-1} = -\frac{K}{k} = -\frac{\lambda}{n\Lambda} \]
m-th order diffractive wave:

- Frequency: \( \omega \pm m\Omega \)
- Diffraction angle: \( \sin \theta_m = m \frac{\lambda}{n\Lambda} \)

\[
J_0(\phi_1) = 0, \quad \phi_1 = \frac{2\pi}{\lambda} \Delta nL \approx 2.405, \ 5.520, \ 8.654, \ldots \quad \text{: zero order is absent}
\]
\[
\phi_1 \approx 1.85 \Rightarrow J_1(\phi_1) = 0.582 \quad \text{: First order diffraction maximum (diffraction efficiency ~ 33.9%)}
\]

\[ L < \frac{\Lambda^2}{2\pi\lambda} \propto \frac{1}{\Omega^2} \]

- The restriction on length of medium is severe at higher frequency
- Diffraction efficiency reduction